Observing Stochastic Processes by Timed Automata

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Workshop on Reachability Problems, Genova, 2011

joint work with Benoît Barbot, Taolue Chen,
Tingting Han and Alexandru Mereacre

September 25, 2011

Let's start easy

Discrete-time Markov chain

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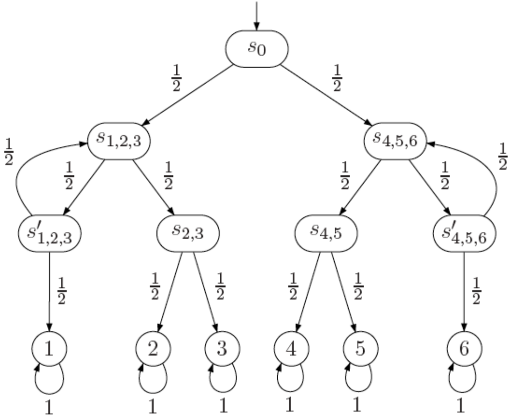
- $\iota_{\text{init}} : S \rightarrow [0, 1]$, the initial distribution with $\sum_{s \in S} \iota_{\text{init}}(s) = 1$

Initial states

- $\iota_{\text{init}}(s)$ is the probability that DTMC $\mathcal{D}$ starts in state s
- the set $\{ s \in S \mid \iota_{\text{init}}(s) > 0 \}$ are the possible initial states.

Simulating a die by a fair coin [Knuth & Yao]



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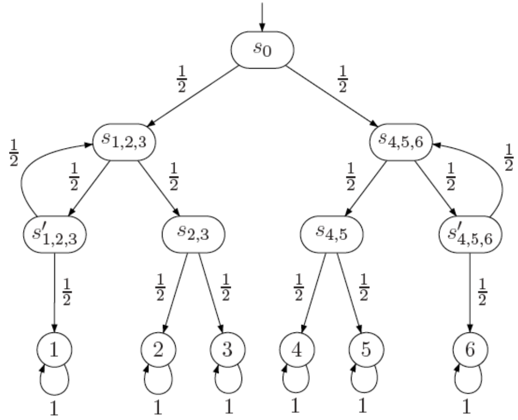

Heads = "go left"; tails = "go right". Does this DTMC adequately model a fair six-sided die?

Some events of interest

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In a similar way, $\Box\Diamond G$ and $\Diamond\Box G$ are defined.

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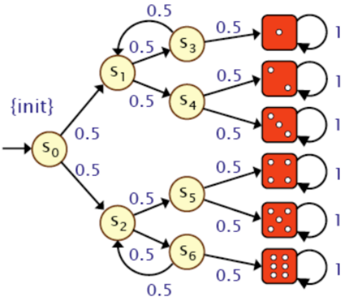
$$x_s = \underbrace{\sum_{t \in S \setminus G} \mathbf{P}(s,t) \cdot x_t}_{\text{reach } G \text{ via } t \in S \setminus G} + \underbrace{\sum_{u \in G} \mathbf{P}(s,u)}_{\text{reach } G \text{ in one step}}$$

Reachability probabilities: Knuth's die



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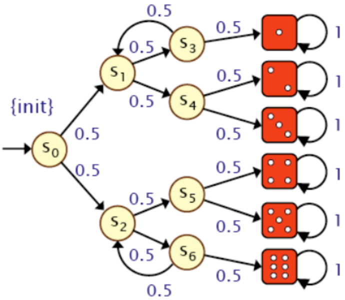

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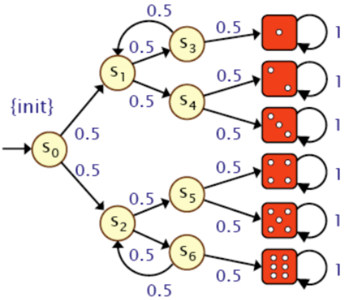

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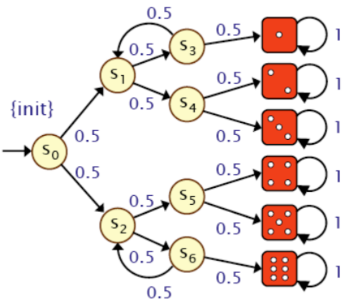

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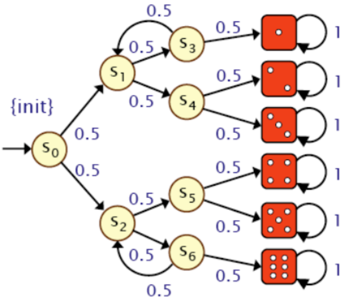

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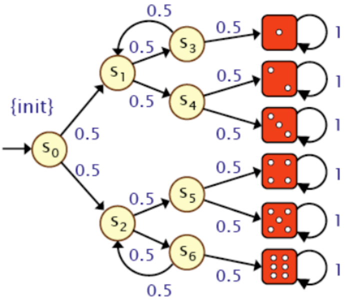

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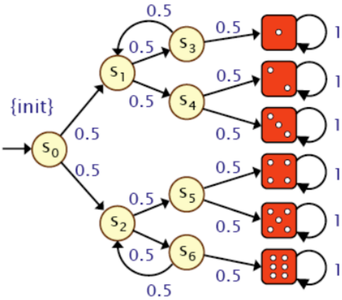

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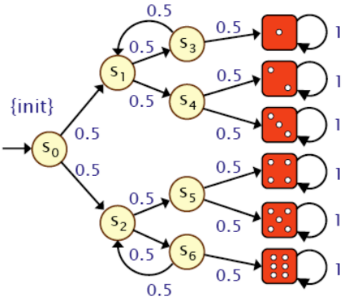

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- Gaussian elimination yields:

$$x_{s_5} = \frac{1}{2},\ x_{s_2} = \frac{1}{3},\ x_{s_6} = \frac{1}{6},\ \text{and } \boxed{x_{s_0} = \frac{1}{6}}$$

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$$\mathbf{x} = \mathbf{A} \cdot \mathbf{x} + \mathbf{b} \quad \text{or} \quad (\mathbf{I} - \mathbf{A}) \cdot \mathbf{x} = \mathbf{b}$$

where $\mathbf{I}$ is the identity matrix of cardinality $|S_?| \times |S_?|$.

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where U is the union of all accepting BSCCs in $\mathcal{D} \otimes \mathcal{A}$. BSCC $T \subseteq S \times Q$ is accepting if $T \cap (S \times L_i) = \varnothing$ and $T \cap (S \times K_i) \neq \varnothing$ for some i.

Synchronous product construction

DTMC $\mathcal{D}$ with state space S

DRA $\mathcal{A}$ with state space Q

$$s_0 \to s_1 \to s_2 \to \cdots \to s_n$$

$$L(s_0) = A_0,\quad L(s_1) = A_1,\quad L(s_2) = A_2,\quad \ldots,\quad L(s_n) = A_n$$

$$q_0 \in Q_0 \xrightarrow{A_0} q_1 \xrightarrow{A_1} q_2 \xrightarrow{A_2} \cdots \to q_n \xrightarrow{A_n} q_{n+1}$$

path

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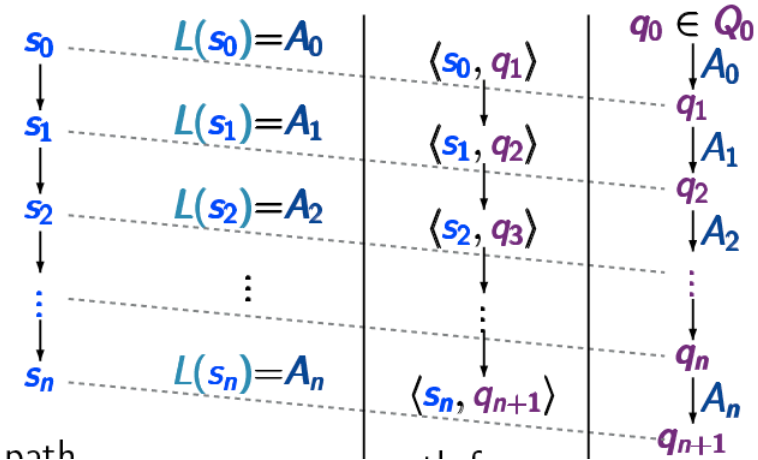

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A graph analysis and solving systems of linear equations suffice.

Random timing

Negative exponential distribution

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Density of exponential distribution

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Variance and expectation

Let r.v. Y be exponentially distributed with rate $\lambda \in \mathbb{R}_{>0}$. Then:

Negative exponential distribution

Density of exponential distribution

The density of an *exponentially distributed* r.v. Y with *rate* $\lambda \in \mathbb{R}_{>0}$ is:

$$f_Y(x) = \lambda \cdot e^{-\lambda \cdot x} \quad \text{for } x > 0 \quad \text{and } f_Y(x) = 0 \text{ otherwise}$$

The cumulative distribution of r.v. Y with rate $\lambda \in \mathbb{R}_{>0}$ is:

$$F_Y(d) = \int_0^d \lambda \cdot e^{-\lambda \cdot x} \, dx = [-e^{-\lambda \cdot x}]_0^d = 1 - e^{-\lambda \cdot d}.$$

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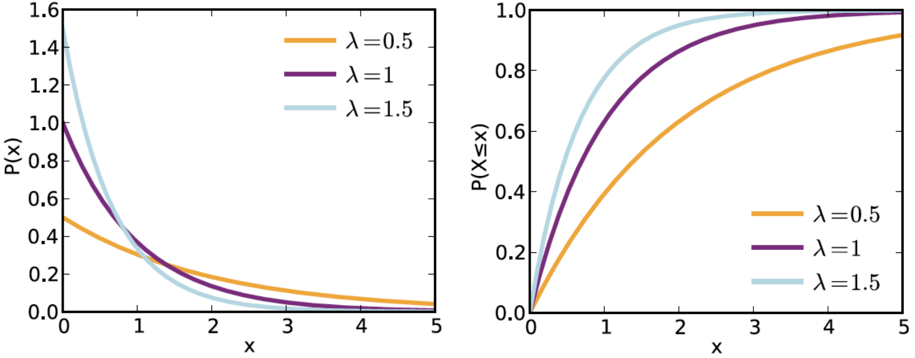
Expectation $E[Y] = \frac{1}{\lambda}$ and variance $Var[Y] = \frac{1}{\lambda^2}$

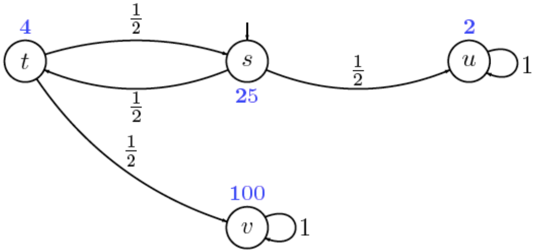
Exponential pdf and cdf



The higher λ, the faster the cdf approaches 1.

Continuous-time Markov chains

A CTMC is a DTMC with an *exit rate* function $r : S \rightarrow \mathbb{R}_{>0}$ where $r(s)$ is the rate of an exponential distribution.



$r(s) = 25$, $r(t) = 4$, $r(u) = 2$ and $r(v) = 100$

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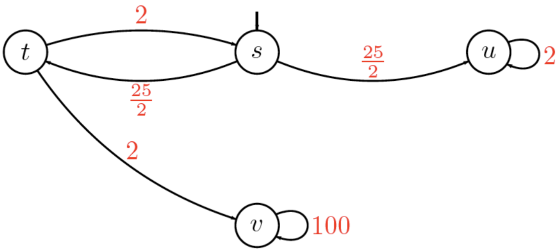

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Residence time distribution

The probability to *take some* outgoing transition from s in $[0, t]$ is:

$$\int_0^t r(s) \cdot e^{-r(s) \cdot x} \, dx \;=\; 1 - e^{-r(s) \cdot t}$$

CTMCs are omnipresent!

- Markovian queueing networks (Kleinrock 1975)
- Stochastic Petri nets (Molloy 1977)
- Stochastic activity networks (Meyer & Sanders 1985)
- Stochastic process algebra (Herzog *et al.*, Hillston 1993)
- Probabilistic input/output automata (Smolka *et al.* 1994)
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CTMCs are one of the most prominent models in performance analysis

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Zeno theorem

[1] Zeno of Elea (490–430 BC), philosopher, famed for his paradoxes.

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Zeno path

Path $s_0 \xrightarrow{t_0} s_1 \xrightarrow{t_1} s_2 \xrightarrow{t_2} s_3 \ldots\ldots$ is called Zeno [1] if $\sum_i t_i$ converges.

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In timed automata, such executions are typically excluded from the analysis.

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For all states s in any CTMC, $Pr\{\, \pi \in Paths(s) \mid \pi \text{ is Zeno} \,\} = 0$.

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Measurability

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Measurability theorem

Events $\Diamond^{I}\, G$, $\Box^{I}\, G$, and $\overline{F}\, \mathrm{U}^{I}\, G$ are measurable on any CTMC.

Timed reachability probabilities in finite CTMCs

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Transient distribution theorem

Theorem: transient distribution as ordinary differential equation

The transient probability vector $\underline{p}(t) = (p_{s_1}(t), \ldots, p_{s_k}(t))$ satisfies:

$$\underline{p}'(t) \;=\; \underline{p}(t) \cdot (\mathbf{R} - \mathbf{r}) \quad \text{given} \quad \underline{p}(0)$$

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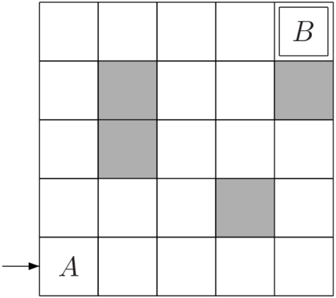
Robot navigation



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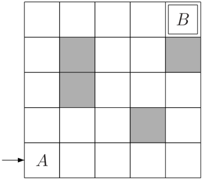

- The robot randomly moves through the cells, and resides in a cell for an exponentially distributed amount of time.
- Gray cells are dangerous; the robot should leave them quickly.

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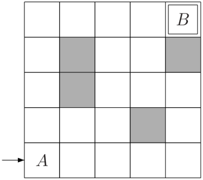

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What is the probability to reach B from A within 10 time units while residing in any dangerous zone for at most 2 time units?

Robot navigation: property

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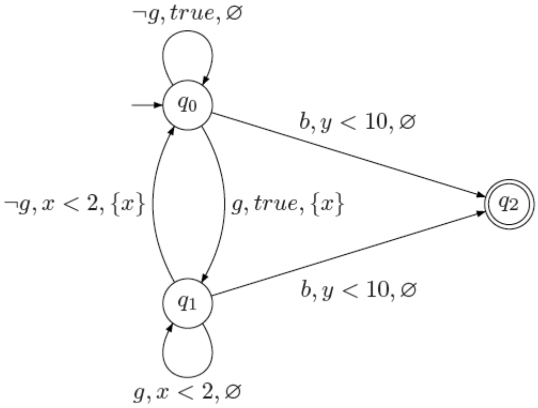
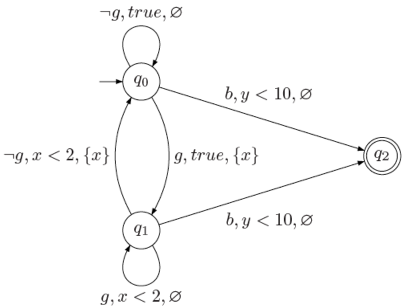


Deterministic timed automata

A Deterministic Timed Automaton (DTA) A is a tuple $(\Sigma, X, Q, q_0, F, \rightarrow)$:



- Σ - *alphabet*
- X - finite set of *clocks*
- Q - finite set of *locations*
- $q_0 \in Q$ - *initial* location
- $F \subseteq Q$ - *accept* locations
- $\rightarrow \in Q \times \Sigma \times \mathcal{C}(X) \times 2^X \times Q$ - *transition relation*;

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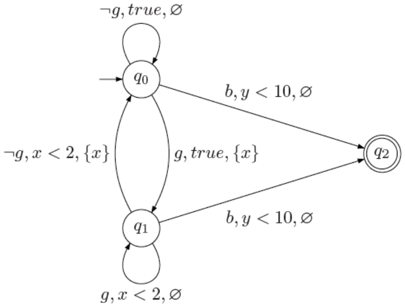

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Determinism: $q \xrightarrow{a,g,X} q'$ and $q \xrightarrow{a,g',X'} q''$ implies $g \cap g' = \varnothing$

What are we interested in?

Problem statement:

Given model CTMC $\mathcal{C}$ and specification DTA $\mathcal{A}$, determine the fraction of runs in $\mathcal{C}$ that satisfy $\mathcal{A}$:

$$Pr(\mathcal{C} \models \mathcal{A}) \;:=\; Pr^{\mathcal{C}}\{\text{Paths in } \mathcal{C} \text{ accepted by } \mathcal{A}\}$$

Theoretical facts

Well-definedness

For any CTMC $\mathcal{C}$ and DTA $\mathcal{A}$, the set $\{\text{Paths in } \mathcal{C} \text{ accepted by } \mathcal{A}\}$ is measurable.

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$Pr(\mathcal{C} \models \mathcal{A})$ equals the probability of accepting paths in $\mathcal{C} \otimes \mathcal{A}$.

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Zone graph construction

1. Reachability probabilities in $\mathcal{C} \otimes \mathcal{A}$ and $ZG(\mathcal{C} \otimes \mathcal{A})$ coincide
2. $ZG(\mathcal{C} \otimes \mathcal{A})$ and $\mathcal{C} \otimes ZG(\mathcal{A})$ are isomorphic
3. $\mathcal{C} \otimes ZG(\mathcal{A})$ is a piecewise-deterministic Markov process [Davis, 1993]

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For any CTMC $\mathcal{C}$ and DTA $\mathcal{A}$, the set $\{$Paths in $\mathcal{C}$ accepted by $\mathcal{A}\}$ is measurable.

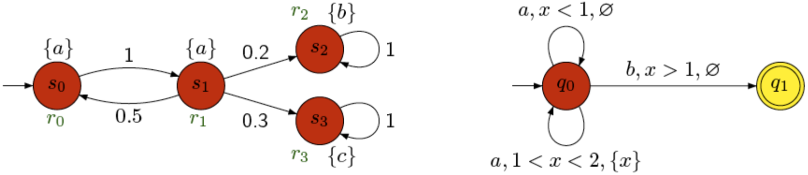
Characterizing the probability of $\mathcal{C} \models \mathcal{A}$ under finite acceptance

$Pr(\mathcal{C} \models \mathcal{A})$ equals the probability of accepting paths in $\mathcal{C} \otimes ZG(\mathcal{A})$.

Characterizing the probability of $\mathcal{C} \models \mathcal{A}$ under Muller acceptance

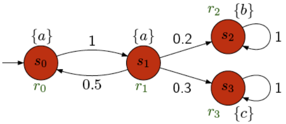
$Pr(\mathcal{C} \models \mathcal{A})$ equals the probability of accepting BSCCs in $\mathcal{C} \otimes ZG(\mathcal{A})$.

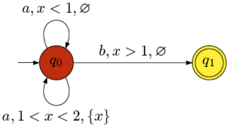
Product construction: example



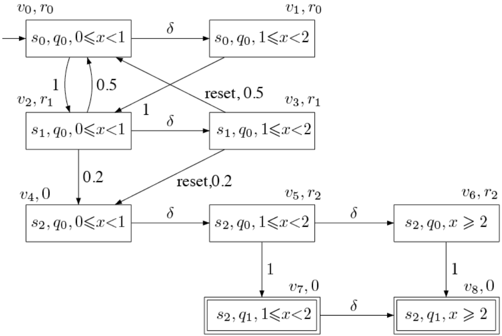
An example CTMC $\mathcal{C}$ (left) and DTA $\mathcal{A}$ (right)

Product construction: example

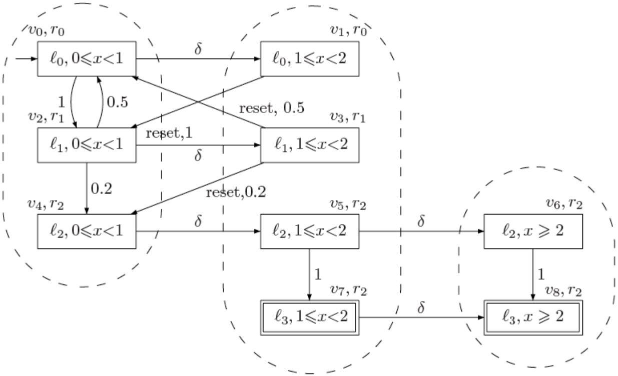




An example CTMC $\mathcal{C}$ (left up) and DTA $\mathcal{A}$ (right up) and $\mathcal{C} \otimes ZG(\mathcal{A})$ (below)



One-clock DTA: partitioning $\mathcal{C} \otimes ZG(\mathcal{A})$



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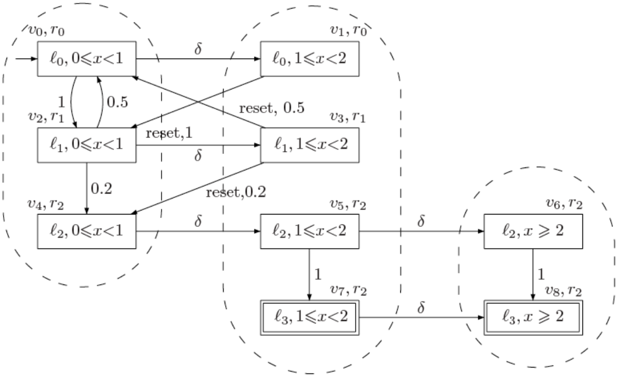

- constants $c_0 < \ldots < c_m$ in A yields $m+1$ subgraphs.
- subgraph i captures behaviour of $\mathcal{C}$ and $\mathcal{A}$ in $[c_i, c_{i+1})$.
- any subgraph is a CTMC, resets lead to subgraph 0, delays to $i+1$.
- a subgraph with its resets yields an "augmented" CTMC.

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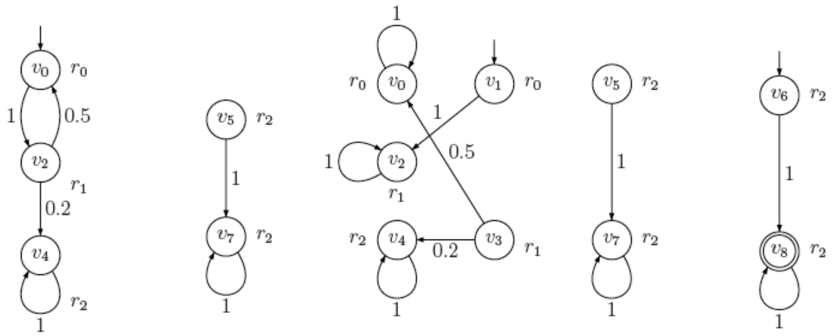

(a) $\mathcal{C}_0$ (b) $\mathcal{C}_1$ (c) $\mathcal{C}_1^a$ (d) $\mathcal{C}_2$

One-clock DTA: characterizing $Pr(\mathcal{C} \models \mathcal{A})$

Theorem

For CTMC $\mathcal{C}$ with initial distribution ι_{init} and 1-clock DTA $\mathcal{A}$ we have:

$$Pr(\mathcal{C} \models \mathcal{A}) \;=\; \iota_{\text{init}} \cdot \mathbf{u}$$

where $\mathbf{u}$ is the solution of the linear equation system $\mathbf{x} \cdot \mathbf{M} = \mathbf{f}$, with

$$\mathbf{M} = \left(\begin{array}{c|c} \mathbf{I}_{n_0} - \mathbf{B}_{m-1} & \mathbf{A}_{m-1} \\ \hline \hat{\mathbf{P}}^{a}_{m} & \mathbf{I}_{n_m} - \mathbf{P}_m \end{array} \right)$$

and $\mathbf{f}$ is the characterizing vector of the final states in subgraph m, and $\mathbf{A}$ and $\mathbf{B}$ are obtained from transient probabilities in all subgraphs.

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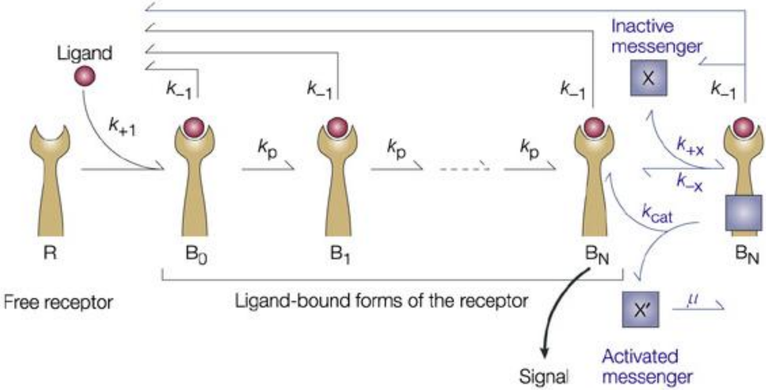
For single-clock DTA, reachability probabilities in (our) PDPs are characterized by the least solution of a linear equation system, whose coefficients are solutions of ODEs (= transient probabilities in CTMCs).

Systems biology: immune-receptor signaling



[Goldstein et. al., Nat. Reviews Immunology, 2004]

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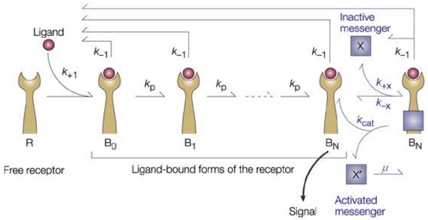

- M ligands can react with a receptor R with rate k_{+1} yielding a ligand-receptor LR
- LR undergoes a sequence of N modifications with a constant rate k_p yielding $B_1, \ldots, B_N$
- LR B_N can link with an inactive messenger with rate k_{+x} yielding a ligand-receptor-messenger (LRM).
- The LRM decomposes into an active messenger with rate k_{cat}

Verification results

M	#CTMC states	No lumping		With lumping			
		# ⊗ states	time(s)	#blocks	time(s)	%transient	%lumping
1	18	31	0	13	0	0%	0%
2	150	203	0.06	56	0.05	58%	39%
3	774	837	1.36	187	0.84	64%	30%
4	3024	2731	17.29	512	9.19	73%	24%
5	9756	7579	152.54	1213	73.4	76%	21%
6	27312	18643	1547.45	2579	457.35	78%	20%
7	68496	41743	11426.46	5038	3185.6	85%	14%
8	157299	86656	23356.5	9200	11950.8	81%	18%
9	336049	169024	71079.15	15906	38637.28	76%	22%
10	675817	312882	205552.36	26256	116314.41	71%	26%

In the case of no lumping, 99% of time is spent on transient analysis

Multi-multi-core model checking

N	4 Cores time(s)	4 Cores speedup	20 Cores time(s)	20 Cores speedup
3	0.45	3.03	0.42	3.22
4	5.3	3.26	3.44	5.02
5	44.73	3.41	15.87	9.61
6	620.16	2.50	160.58	9.64
7	4142.19	2.76	949.32	12.04
8	8168.62	2.86	1722.63	13.56
9	23865.17	2.98	5457.01	13.03
10	70623.46	2.91	16699.22	12.31

Parallelization of the transient analysis only; not the lumping.

Non-determinism: MDP

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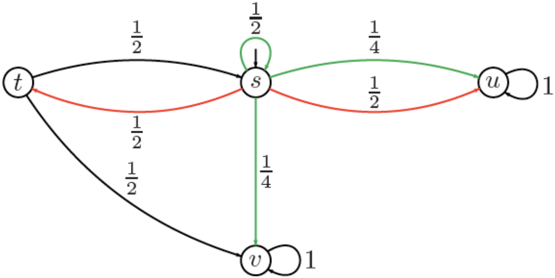
An MDP is a DTMC in which in any state a non-deterministic choice between probability distributions exists.



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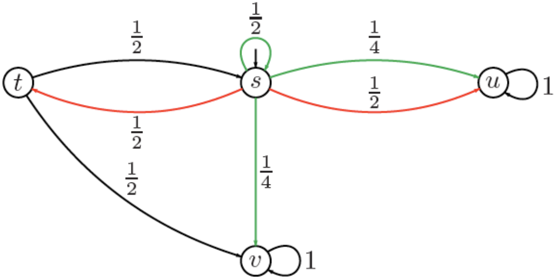

Set of enabled distributions (= colors) in state s is $Act(s) = \{ \alpha, \beta \}$ where

- $\mathbf{P}(s, \alpha, s) = \frac{1}{2}$, $\mathbf{P}(s, \alpha, t) = 0$ and $\mathbf{P}(s, \alpha, u) = \mathbf{P}(s, \alpha, v) = \frac{1}{4}$
- $\mathbf{P}(s, \beta, s) = \mathbf{P}(s, \beta, v) = 0$, and $\mathbf{P}(s, \beta, t) = \mathbf{P}(s, \beta, u) = \frac{1}{2}$

Continuous-time Markov decision processes

A CTMDP is an MDP with an *exit rate* function $r : S \times Act \rightarrow \mathbb{R}_{>0}$ where $r(s, \alpha)$ is the rate of an exponential distribution.

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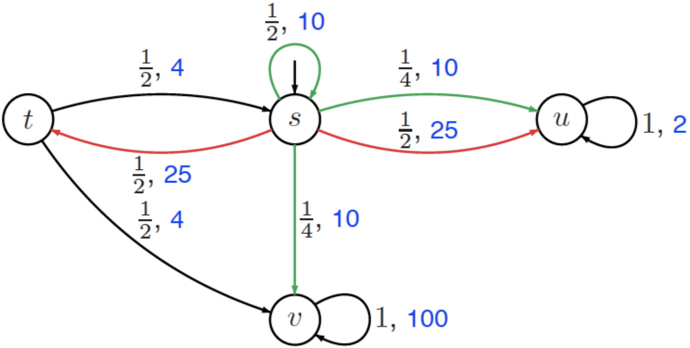

$r(s, \alpha) = 10$ and $r(s, \beta) = 25$

Timed reachability objectives

Policy

Non-determinism is reduced by a policy.

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Timed reachability

Let $G \subseteq S$ be a finite set of goal states and $t \in \mathbb{R}_{\geqslant 0}$ a deadline.
Time-bounded reachability probability from state s under policy $\mathfrak{S}$:

$$Pr^{\mathfrak{S}}(s \models \Diamond^{\leqslant t} G) \;=\; Pr_s^{\mathcal{C}_{\mathfrak{S}}}\{\, \pi \in Paths(s) \mid \pi \models \Diamond^{\leqslant t} G \,\}$$

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Analysis focuses on obtaining lower- and upperbounds, e.g.,

$$Pr^{\max}(s \models \Diamond^{\leqslant t} G) \;=\; \sup_{\mathfrak{S}} Pr^{\mathfrak{S}}(s \models \Diamond^{\leqslant t} G)$$

where $\mathfrak{S}$ ranges over all possible policies.

Maximal timed reachability

Characterisation of timed reachability probabilities

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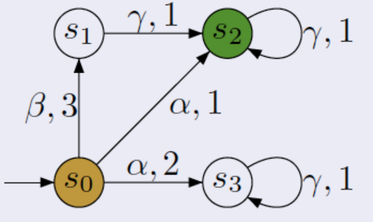
- For any state $s \in Pre^*(G) \setminus G$:

$$x_s(t) = \max_{\alpha \in Act(s)} \int_0^t \sum_{s' \in S} \underbrace{\mathbf{R}(s, \alpha, s') \cdot e^{-r(s,\alpha)\cdot x}}_{\substack{\text{probability to move to}\\ \text{state } s' \text{ at time } x\\ \text{under action } \alpha}} \cdot \underbrace{x_{s'}(t-x)}_{\substack{\text{max. prob.}\\ \text{to fulfill } \Diamond^{\leqslant t-x}\, G\\ \text{from } s'}} dx$$

Timed policies are optimal



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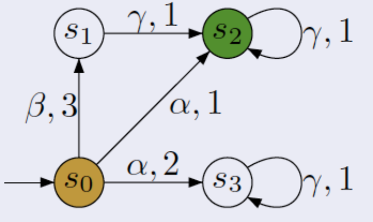

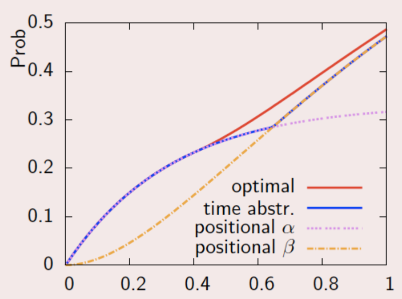

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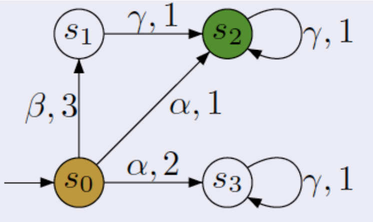

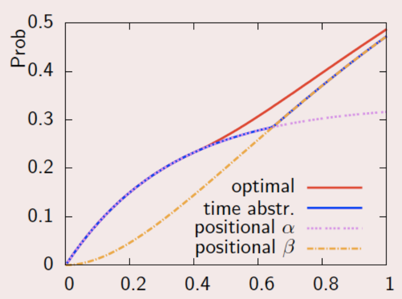

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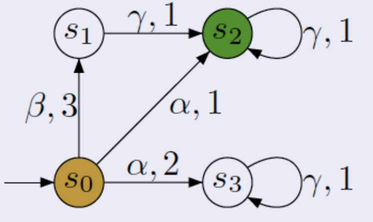

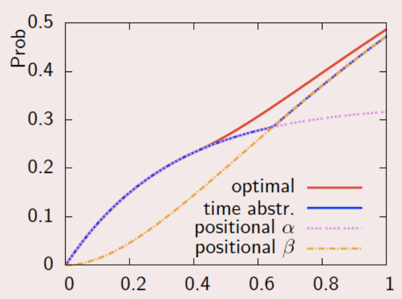

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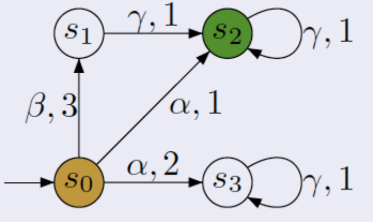

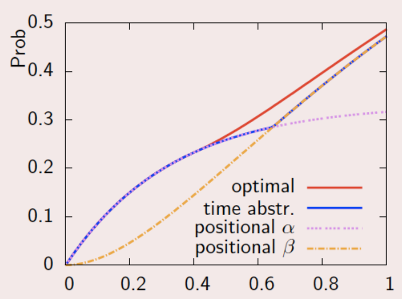

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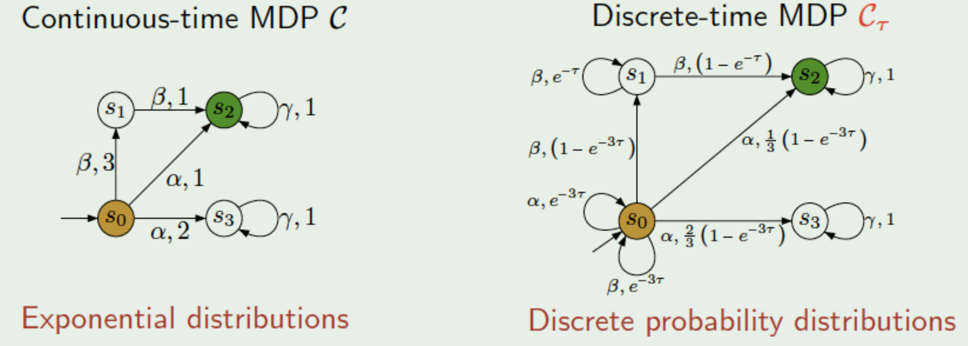
- If long time remains: choose β; if short time remains: choose α.
- Optimal policy for $t{=}1$: choose α if $1{-}t_0 \leqslant \ln 3 - \ln 2$, otherwise β

Discretisation



Reachability in d time $\approx$ Reachability in $\frac{d}{\tau}$ steps

Checking CTMDPs against DTA objectives

Problem statement:

Given model CTMDP $\mathcal{C}$ and specification DTA $\mathcal{A}$, determine the maximal fraction of runs in $\mathcal{C}$ that satisfying $\mathcal{A}$:

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One-clock DTA: characterizing $Pr^{\max}(\mathcal{C} \models \mathcal{A})$

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Verifying a CTMC against a 1-clock DTA

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$Pr^{\max}(\mathcal{C} \models \mathcal{A})$ can be characterised as the unique solution of a linear inequation system whose coefficients are maximal timed reachability probabilities in CTMDP $\mathcal{C}$.

For details, please consult the paper in the RP'11 proceedings.

Related work

- Observers for timed automata (Aceto *et al.* JLAP 2003)
- Timed automata for GSMPs (Brazdil *et al.* HSCC 2011)
- PTCTL model checking of PTA (Kwiatkowska *et el.* TCS 2002)
- CSL with regular expressions (Baier *et al.* IEEE TSE 2007)
- CSL with one-clock DTA as time constraints (Donatelli *et al.* IEEE TSE 2009)
  - for single-clock DTA, our results coincide
  - . . . but we obtain the results in a different manner
- Probabilistic semantics of TA (Baier *et al.* LICS 2008)

Epilogue

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Take-home messages

- Timed reachability in a CTMC $\mathcal{C}$ = transient analysis of $\mathcal{C}$

Epilogue

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- For CTMDPs: similar approach using linear inequations.
- Prototypical tool-support for 1-clock DTA (to be in PRISM).